PRACTICE FINAL − SOLUTIONS

PEYAM RYAN TABRIZIAN

1. State carefully:

(a) The Fundamental Theorem of Calculus, part 1

If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \leq x \leq b$$

is continuous on $[a,b]$, and differentiable on (a,b), and $g'(x) = f(x)$

(b) The Fundamental Theorem of Calculus, part 2

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$

Date: Friday, December 6th, 2013.

2. Prove directly from the definition that $\int_0^1 4x dx = 2$
(Recall that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$)

$f(x) = 4x$, $a = 0$, $b = 1$

$\Delta x = \frac{b-a}{n} = \frac{1}{n}$, and $x_i = a + i\Delta x = \frac{i}{n}$

Hence:

$$\begin{aligned}\int_0^1 4x dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^n 4\left(\frac{i}{n}\right)\frac{1}{n} \\ &= \lim_{n\to\infty} \sum_{i=1}^n \frac{4i}{n^2} \\ &= \lim_{n\to\infty} \frac{4}{n^2}\sum_{i=1}^n i \\ &= \lim_{n\to\infty} \frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) \\ &= \lim_{n\to\infty} \frac{2(n+1)}{n} \\ &=2\end{aligned}$$

3. Water flows into a tank, the inflow rate at time t hours being $r(t) = te^{-t^2}$ cubic meters per hour. How much water flows into the tank between times $t = 1$ and $t = 2$?

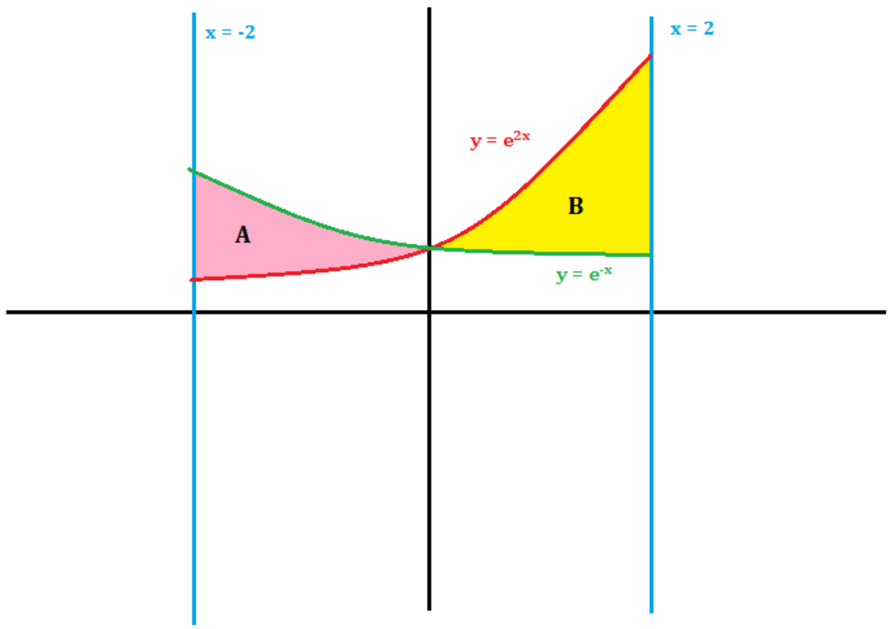
Recall that the net change is the integral of the rate of change, so our answer is:

$$
\begin{aligned}
A &= \int_1^2 r(t)dt \\
&= \int_1^2 te^{-t^2} dt \\
&= \int_{-1}^{-4} e^u \left(-\frac{1}{2}\right) du \qquad \text{using the substitution } u = -t^2 \\
&= -\frac{1}{2}\int_{-1}^{-4} e^u du \\
&= -\frac{1}{2}\left[e^u\right]_{-1}^{-4} \\
&= -\frac{1}{2}(e^{-4} - e^{-1}) \\
&= \frac{1}{2}(e^{-1} - e^{-4})
\end{aligned}
$$

4. Find the area of the region bounded by the curves $y = e^{-x}$, $y = e^{2x}$, $x = -2$, and $x = 2$

1) Picture:

1A/Practice Exams/Steelarea.png



(Notice how the use of colored pencils makes the picture clearer! Bring colored pencils to the exam!)

2) Points of intersection:

$$e^{2x} = e^{-x} \Leftrightarrow e^{3x} = 1 \Leftrightarrow 3x = 0 \Leftrightarrow x = 0$$

3) Now from the picture, it is clear that the total area is $A + B$. But from the picture, we have:

$$A = \int_{-2}^{0} e^{-x} - e^{2x} dx = \left[-e^{-x} - \frac{1}{2} e^{2x} \right]_{-2}^{0} = -1 - \frac{1}{2} + e^2 + \frac{1}{2} e^{-4} = -\frac{3}{2} + e^2 + \frac{1}{2} e^{-4}$$

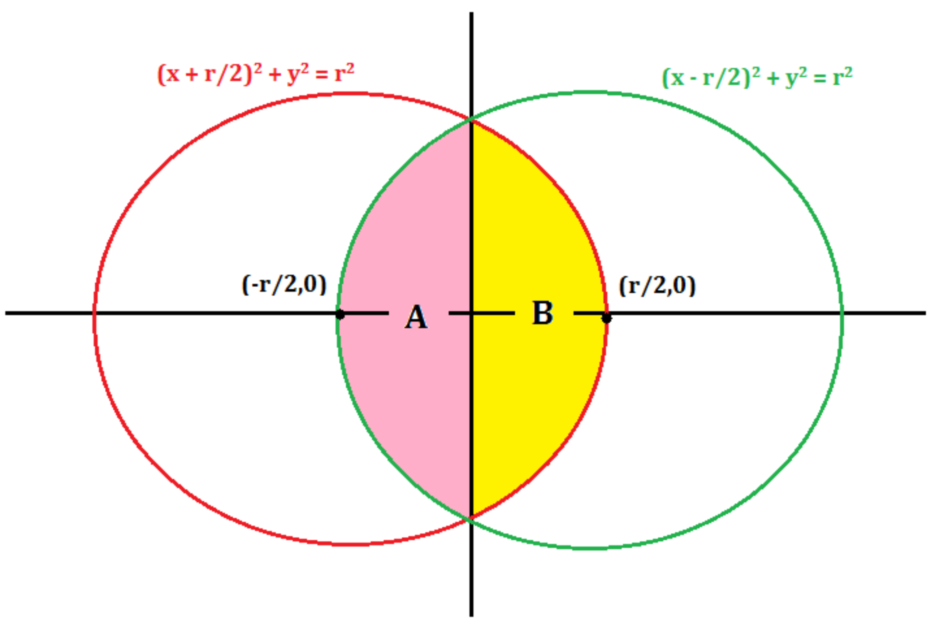
$$B = \int_{0}^{2} e^{2x} - e^{-x} dx = \left[\frac{1}{2} e^{2x} + e^{-x} \right]_{0}^{2} = \frac{1}{2} e^4 + e^{-2} - \frac{1}{2} - 1 = -\frac{3}{2} + \frac{1}{2} e^4 + e^{-2}$$

Hence $\boxed{\text{Area} = A + B = -3 + e^2 + \frac{e^{-4}}{2} + \frac{e^4}{2} + e^{-2}}$

5. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other.

The trick is to place your spheres correctly! Make one of them have center $(-\frac{r}{2}, 0)$ and the other one have center $(\frac{r}{2}, 0)$ in the $xy-$ plane, as in the following picture (the picture is a 2D profile-view of the 3D-situation):

1A/Practice Exams/Steelspheres.png



First of all, notice the symmetry! The he volume of A rotated about the x-axis (call it V^-) **equals** to the volume of B rotated about the x-axis (call it V^+), i.e. $\boxed{V^+ = V^-}$. So the total volume is $V = V^- + V^+ = 2V^+$.

Calculation of V^+

Using the disk method, the endpoints are $a = 0$ and $b = \frac{r}{2}$, and from the equation of the **red** circle (again, notice how using colors helps you simplify the problem), we get $(x + \frac{r}{2})^2 + y^2 = r^2$, so $y^2 = r^2 - (x + \frac{r}{2})^2$.

Hence:

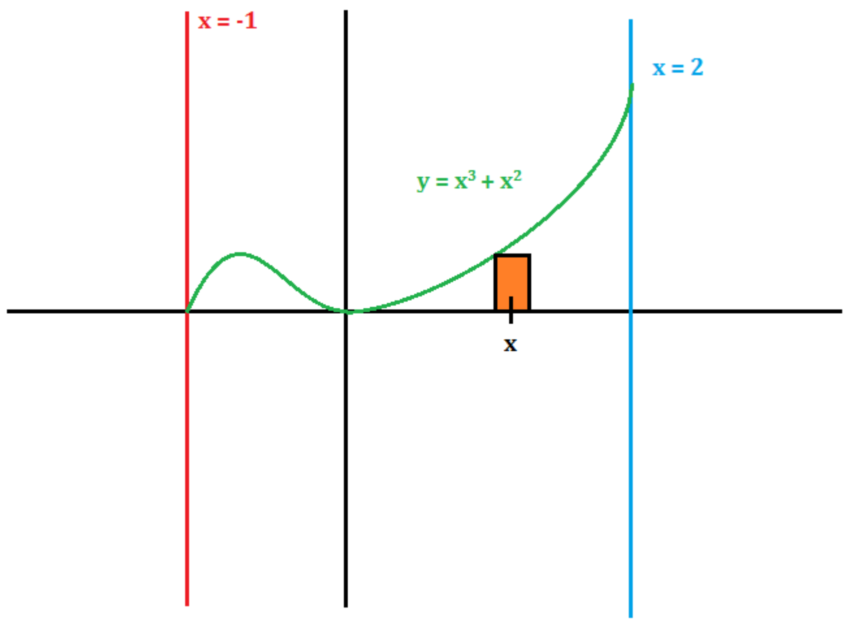
$$\begin{aligned}
V^+ &= \int_0^{\frac{r}{2}} \pi y^2 dx \\
&= \int_0^{\frac{r}{2}} \pi \left(r^2 - (x + \frac{r}{2})^2 \right) dx \\
&= \pi \left[r^2 x - \frac{1}{3}(x + \frac{r}{2})^3 \right]_0^{\frac{r}{2}} \\
&= \pi \left(\frac{r^3}{2} - \frac{r^3}{3} - 0 + \frac{1}{3}(\frac{r}{2})^3 \right) \\
&= \pi \left(\frac{r^3}{2} - \frac{r^3}{3} + \frac{r^3}{24} \right) \\
&= \frac{5\pi r^3}{24}
\end{aligned}$$

So $\boxed{V = 2V^+ = \frac{5}{12}\pi r^3}$

6. The region bounded by the curves $y = x^3 + x^2$, $x = 2$, and the $x-$axis is rotated about the line $x = -1$. What is the volume of the resulting solid?

Picture:

1A/Practice Exams/Steelshell.png



Note: To help draw the picture, it might help to notice that $x^3+x^2 = x^2(x+1)$.

Now the disk method doesn't work because the slices are not disks, and the washer method doesn't work either because you'd have to solve for x in terms of y. Hence, we need to use the **shell** method.

The radius is $x - (-1) = x + 1$ and the height of a typical shell is $x^3 + x^2$, whence the volume is:

$$
\begin{aligned}
V = & \int_{-1}^{2} 2\pi(x+1)(x^3+x^2)dx \\
= & 2\pi \int_{-1}^{2} x^4 + x^3 + x^3 + x^2 dx \\
= & 2\pi \int_{-1}^{2} x^4 + 2x^3 + x^2 dx \\
= & 2\pi \left[\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} \right]_{-1}^{2} \\
= & 2\pi \left(\frac{32}{5} + 8 + \frac{8}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{3} \right) \\
= & \frac{171\pi}{5}
\end{aligned}
$$

7. Compute $\int_0^{\frac{2}{3}} \frac{1}{4+9x^2}dx$

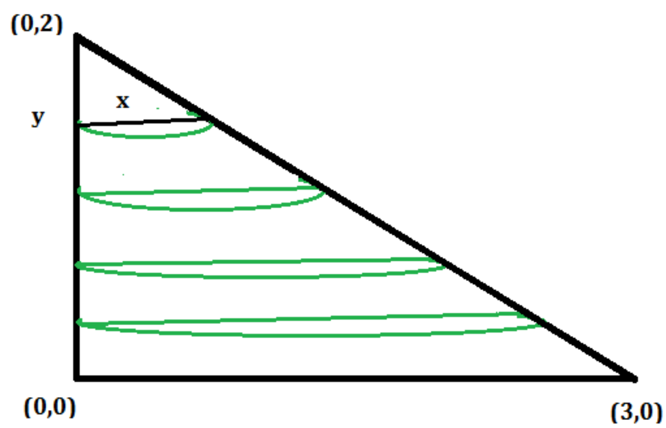
$$\begin{aligned}
\int_0^{\frac{2}{3}} \frac{1}{4+9x^2}dx &= \int_0^{\frac{2}{3}} \frac{1}{4}\frac{1}{1+\frac{9}{4}x^2}dx \\
&= \frac{1}{4}\int_0^{\frac{2}{3}} \frac{1}{1+\left(\frac{3x}{2}\right)^2}dx \\
&= \frac{1}{4}\int_0^{1} \frac{1}{1+u^2}\frac{2}{3}du \qquad \text{using the substitution } u = \frac{3x}{2} \\
&= \frac{1}{6}(\tan^{-1}(1) - \tan^{-1}(0)) \\
&= \frac{1}{6}(\frac{\pi}{4} - 0) \\
&= \frac{\pi}{24}
\end{aligned}$$

8. The base of a solid is the triangular region with vertices $(0,0)$, $(3,0)$, and $(0,2)$. Its cross-sections perpendicular to the $y-$axis are semicircles. What is its volume?

The tricky thing about this problem is that none of the usual methods (disk, washer, shell) work because the solid is **not** a solid of revolution so we have to resort to more primitive techniques, namely $\boxed{V = \int_0^2 A(y)dy}$ (we get the endpoints 0 and 2 because $0 \leq y \leq 2$)

Now the picture is as follows:

1A/Practice Exams/Steeltriangle.png

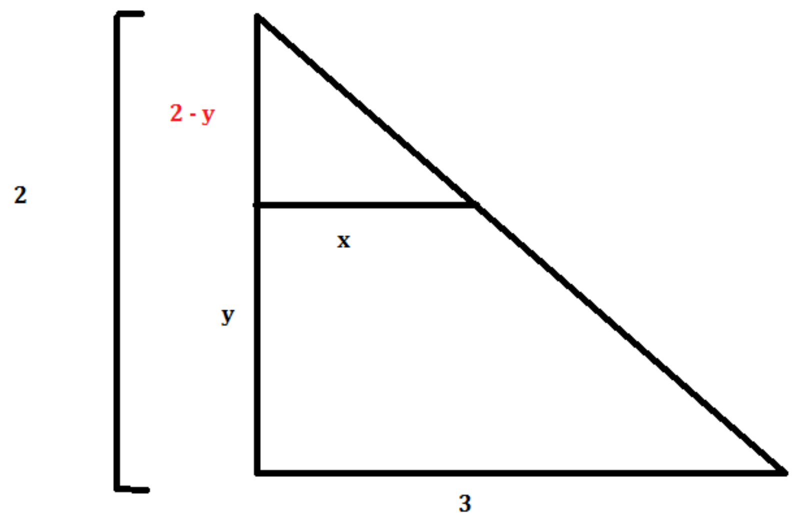


Here we're given that the slices are semi-circles, so $A(y) = \frac{\pi}{2}\left(\frac{x}{2}\right)^2$, where x is the **diameter** of the semi-circle, as in the picture above!

Now we need to calculate x in terms of y!

Profile-view of the picture above

1A/Practice Exams/Steeltriangleprofile.png



Using the law of similar triangles, we get:

$$\frac{2-y}{2} = \frac{x}{3}$$

So $x = 3\left(\frac{2-y}{2}\right) = 3 - \frac{3}{2}y$.
And finally:

$$\begin{aligned}
V &= \int_0^2 A(y)dy \\
&= \int_0^2 \frac{\pi}{2}\left(\frac{x}{2}\right)^2 dy \\
&= \int_0^2 \frac{\pi}{2}\left(\frac{3}{2} - \frac{3}{4}y\right)^2 dy \\
&= \frac{\pi}{2}\int_0^2 \frac{9}{16}y^2 - \frac{9}{4}y + \frac{9}{4}dy \\
&= \frac{9\pi}{8}\int_0^2 \frac{y^2}{4} - y + 1dy \\
&= \frac{9\pi}{8}\left[\frac{y^3}{12} - \frac{y^2}{2} + y\right]_0^2 \\
&= \frac{9\pi}{8}\left(\frac{2}{3} - 2 + 2 - 0 + 0 - 0\right) \\
&= \frac{3\pi}{4}
\end{aligned}$$